

Introduction to Transverse Lattice Gauge Theory¹

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Abstract. I review a new treatment of an old idea for light-front quantization of lattice gauge theories and give new results from some illustrative calculations: [I] Transverse Lattice Gauge Theory; [II] Pure Glue; [III] Heavy Sources and Winding Modes; [IV] An Example – Large- N QCD in $2 + 1$ Dimensions.

I TRANSVERSE LATTICE GAUGE THEORY

A Light-Front Co-ordinates

In the constituent quark model, hadrons are composed of a few quarks moving about in empty space and bound by non-relativistic potentials. It is an *extremely* successful model. Before the advent of QCD as the quantum field theory of hadron structure, there was little reason to think of hadrons in any other way. But while there is consensus that QCD is fundamentally correct, the low-energy dynamics of this gauge theory are thought to involve a complicated vacuum with arbitrary numbers of relativistic quarks and gluons. How could these pictures be reconciled? Indeed, how can one efficiently tackle the problem of relativistic strongly bound states at all?

There is a popular yet poorly understood formulation of quantum field theory — Light-Front quantisation — where something like the constituent picture of boundstates arises naturally. In fact, it offers much more than that, since it furnishes a Schrodinger equation suitable for relativistic many-body systems.

In the co-ordinate system $\{x^0, x^1, x^2, x^3\}$, where x^0 is time (I set $c = \hbar = 1$), one usually initializes the wavefunction $\Psi(x^1, x^2, x^3)$ on a hyper-plane $x^0 =$

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constant. If the four-momentum of the system is $\{P^0, P^1, P^2, P^3\}$, the energy-operator \hat{P}^0 is the equal- x^0 hamiltonian that evolves the wavefunction in x^0 according to Shrodinger's equation

$$i\frac{\partial\Psi}{\partial x^0} = \hat{P}^0\Psi. \quad (1)$$

This formulation has manifest rotational invariance and is tractable for most non-relativistic problems in physics. It is not so suitable when the momenta $P^{i=1,2,3} \sim P^0$ (recall that P^0 contains the rest-energy).

In light-front (LF) quantisation [1] on the other hand, we (conventionally) define

$$x^\pm = \frac{x^0 \pm x^3}{\sqrt{2}}, \quad P^\pm = \frac{P^0 \pm P^3}{\sqrt{2}}. \quad (2)$$

P^\pm is canonically conjugate to x^\mp , and we interpret (x^-, P^+) as space and momentum, while (x^+, P^-) as time and energy. $\mathbf{x} = \{x^1, x^2\}$ and $\mathbf{P} = \{P^1, P^2\}$ are the transverse co-ordinates. The Minkowski metric in the new co-ordinate system is such that the invariant length

$$ds^2 = 2dx^+dx^- - (dx^1)^2 - (dx^2)^2 \quad (3)$$

and $x^+ \equiv x_-$. The wavefunction Ψ_{lc} is initialized on a null hyper-plane $x^+ = \text{constant}$ and the LF Schrodinger equation that evolves $\Psi_{lc}(x^-, x^1, x^2)$ in LF time x^+ is

$$i\frac{\partial\Psi_{lc}}{\partial x^+} = \hat{P}^-\Psi_{lc}. \quad (4)$$

This equation is now suitable for relativistic problems. For example, Ψ_{lc} is manifestly Lorentz-boost invariant; a detailed discussion of the advantages and historical survey of LF ideas can be found in ref. [2].

It is easy to see why LF co-ordinates may help with the constituent question, if we consider the energy-momentum relation for a (free) particle of mass μ

$$P^- = \frac{(P^1)^2 + (P^2)^2 + \mu^2}{2P^+}. \quad (5)$$

With the conventional interpretation of positive energy ($P^0 \geq 0$) for particles and anti-particles, it follows that light-front momentum $P^+ \geq 0$. Because P^+ is conserved and the total P^+ of the vacuum must be zero (by translation invariance), each particle contributing to the vacuum state can have only $P^+ = 0$. But according to (5), these 'zero modes' have infinite light-front energy. If we introduce a high energy cut-off, as is always necessary for defining a quantum field theory, these zero modes and the vacuum structure they carry will be explicitly removed. Their effects on observables below the cut-off

should be felt through renormalisation of the hamiltonian. If this renormalisation entails quarks getting a constituent mass, and the gluon degrees of freedom getting at least a mass gap, if not constituent masses themselves, then we have arrived at the basic picture of the constituent quark model.

The trouble with the above idea is that the LF renormalisation procedure is still quite poorly understood for theories like QCD, though interesting efforts are being made in this direction [8–11]. A formulation of quantum field theory where non-perturbative effects normally associated with the vacuum have to appear via explicit counter-terms in the hamiltonian is also alien to many people. For these reasons there have been some reservations about the consistency of light-front quantisation. Nevertheless, the ideas are physically appealing and a number of examples are known where one can derive the renormalisation resulting from the removal of certain non-dynamical zero modes; see Burkardt’s lectures at this school in a previous year [12]. The attitude I will take in these lectures is that the potential applications of light-front quantisation are far too important for it to be simply dismissed. I will describe a practical framework for doing calculations in LF QCD with a high-energy cut-off, pioneered by Bardeen and Pearson [13,14], which uses symmetry to guide renormalisation of the theory. The symmetries that define QCD are gauge, Lorentz, and chiral invariance. In these lectures, I will only discuss pure gauge theories and infinitely heavy quarks, so chiral invariance will not be treated.

The method will initially allow all operators in the LF hamiltonian that respect symmetries unviolated by the cut-off — allowed operators. After systematically truncating to a subset of these allowed operators, one then tunes the remaining couplings *a posteriori* to restore the violated symmetries in observables as best one can. This is a physically motivated, systematically improvable framework which does not use any fits to experimental data, i.e. it uses only first principles. At present, the tests for violations of Lorentz covariance are done in a rather inefficient way, simply by solving a whole bunch of hamiltonians to find the correct one. Nevertheless, the recent results obtained in this way [5–7] are surprisingly accurate. Once a more efficient treatment of Lorentz covariance is developed, or non-perturbative LF hamiltonian renormalisation group ideas are better understood, many new areas of non-perturbative QCD will be opened up.

B Transverse Lattices

Light-front quantisation already has more manifest Lorentz symmetry than any other hamiltonian quantisation scheme [1], but typically one cannot avoid breaking rotational invariance; the choice x^3 in (2), rather than x^1 or x^2 , is arbitrary. On the other hand, we would like to preserve gauge invariance (later this will also greatly facilitate the treatment of confinement). This

can be done with a lattice cut-off [15,16]. In a hamiltonian formulation time remains continous and infinite, of course. For a LF hamiltonian \hat{P}^- , whose configuration space is $\{x^-, x^1, x^2\}$, only lattice discretization of the transverse directions \mathbf{x} is appropriate. Discretizing the longitudinal co-ordinate x^- is not appropriate since it would cut-off large values of the conjugate variable P^+ ; but it's small values of P^+ that correspond to high energy. To remove the high-energy region in a gauge-invariant way we can impose, say, anti-periodic boundary conditions on x^- [17]. In the following we use indices, $\mu, \nu \in \{0, 1, 2, 3\}$, $\alpha, \beta \in \{+, -\}$ and $r \in \{1, 2\}$, and sum over repeated indices. We introduce a transverse lattice spacing a and a longitudinal period \mathcal{L} .

To construct an $SU(N)$ transverse lattice (pure) gauge theory on this space-time, we introduce gauge potentials A_α in the Lie algebra of $SU(N)$ for the continuum directions, and lattice variables for the transverse directions. The particular choice of lattice variables which will be convenient for LF quantisation are complex $N \times N$ matrices $M_r(x^+, x^-, \mathbf{x})$ associated with the transverse link from \mathbf{x} to $\mathbf{x} + a\hat{\mathbf{r}}$, at position (x^+, x^-) — colour-dielectric link variables. Because they are linear variables, it is easy to identify the independent degrees of freedom. It is sometimes helpful to think of them as an average over paths \mathcal{C} between these points, with some weight $\rho(\mathcal{C})$, of the short-distance continuum gauge potentials

$$M_r = \sum_{\mathcal{C}} \rho(\mathcal{C}) \text{P exp} \left\{ i \int_{\mathbf{x}}^{\mathbf{x}+a\hat{\mathbf{r}}} A_\mu dx^\mu \right\} \quad (6)$$

Near the continuum limit $a = 0$, the potentials must change only very slowly over many lattice spacings, and M must be forced to lie in the $SU(N)$ group. At larger a there is no such restriction however. In fact, for non-abelian gauge theories, it makes more physical sense to use disordered complex variables M when the lattice cut-off a is quite large.

Continuum gauge transformations induce the following lattice gauge transformations on these variables

$$A_\alpha(\mathbf{x}) \rightarrow V(\mathbf{x}) A_\alpha(\mathbf{x}) V^\dagger(\mathbf{x}) + i (\partial_\alpha V(\mathbf{x})) V^\dagger(\mathbf{x}) , \quad (7)$$

$$M_r(\mathbf{x}) \rightarrow V(\mathbf{x}) M_r(\mathbf{x}) V^\dagger(\mathbf{x} + a\hat{\mathbf{r}}) , \quad (8)$$

where $V \in SU(N)$. The strategy will be to construct LF hamiltonians invariant under these gauge transformations and any Lorentz symmetries that have not be violated by the cut-offs; this include boosts along x^3 and discrete Z_4 rotations about x^3 . We will seek to enhance the remaining Lorentz symmetries, generically broken by the cut-offs, by tuning the couplings of these hamiltonians.

The continuum QCD action is

$$- \int d^4x \frac{1}{2g^2} \text{Tr} \{ F_{\mu\nu} F^{\mu\nu} \} . \quad (9)$$

The following gauge-invariant transverse lattice action is the simplest that reduces to (9) in the naive continuum limit $a \rightarrow 0$, $\mathcal{L} \rightarrow \infty$, [14]

$$S = \int dx^+ dx^- \sum_{\mathbf{x}} \text{Tr} \left\{ \bar{D}_\alpha M_r(\mathbf{x}) \left(\bar{D}^\alpha M_r(\mathbf{x}) \right)^\dagger \right\} - \frac{a^2}{2g^2} \text{Tr} \{ F_{\alpha\beta} F^{\alpha\beta} \} + \frac{\beta}{Na^2} \text{Tr} \{ M_{\text{plaq}} + M_{\text{plaq}}^\dagger \} - U[M] \quad (10)$$

where

$$\bar{D}_\alpha M_r(\mathbf{x}) = (\partial_\alpha + iA_\alpha(\mathbf{x})) M_r(\mathbf{x}) - iM_r(\mathbf{x}) A_\alpha(\mathbf{x} + a\hat{\mathbf{r}}), \quad (11)$$

M_{plaq} is the product of link matrices around a transverse plaquette, and $U[M]$ is any gauge-invariant potential that forces M into $SU(N)$ as $a \rightarrow 0$, such as

$$U[M] = \frac{N}{\lambda} \left(\text{Tr} \{ (1 - M_r^\dagger(\mathbf{x}) M_r(\mathbf{x}))^2 \} + (\det M - 1)^2 \right), \quad (12)$$

where $\lambda \rightarrow 0$ as $a \rightarrow 0$.

More generally, $S[M]$ can consist of all the combinations of $M, F^{\alpha\beta}, \bar{D}^\alpha M$ which are invariant under gauge and residual Lorentz symmetries. Their couplings are to be chosen so as to restore the symmetries violated by the cut-offs.

II PURE GLUE

A Colour-Dielectric Expansion

$S[M]$ contains an infinite number of allowed operators, so to begin calculations we need to truncate them to a finite set in some physically reasonable way. Of the two cut-offs, a and \mathcal{L} , it will be possible to extrapolate the latter. It makes sense therefore to restrict allowed operators on dimensional grounds w.r.t. $\{0, 3\}$ co-ordinates. The divergences that appear as $\mathcal{L} \rightarrow \infty$ are of normal-ordering type, and thus easily dealt with. We will also demand an interaction-independent (kinetic) momentum operator P^+ and naive LF parity $x^+ \leftrightarrow x^-$. The main further restriction will be to expand the LF hamiltonian \hat{P}^- , that results from $S[M]$, in powers of M — the ‘colour-dielectric expansion’. This only makes sense if, for a given a , the P^- that best recovers Lorentz covariance is analytic about $M = 0$ and (classically) minimized there. (This could not be the case in a neighborhood of $a = 0$, where M should be forced into $SU(N)$). We are not guaranteed to find anything at all. There is, however, circumstantial evidence from Euclidean lattice work [18] that a Lorentz covariant scaling trajectory in the space of couplings flows from the continuum limit to a large a region where $M = 0$ is the classical minimum. Our job is to explicitly verify the existence of such a trajectory for the transverse lattice by testing this region for signs of Lorentz covariance restoration.

B Gauge-Fixing

Light-front quantization is greatly simplified by the LF gauge choice $A_- = 0$.² This still leaves x^- -independent gauge transformations unfixed. Suppose we start with an action

$$S = \int dx^0 dx^3 \sum_{\mathbf{x}} \left(\text{Tr} \left\{ \bar{D}_\alpha M_r(\mathbf{x}) \left(\bar{D}^\alpha M_r(\mathbf{x}) \right)^\dagger \right\} - \frac{1}{2G^2} \text{Tr} \left\{ F_{\alpha\beta} F^{\alpha\beta} \right\} - V_{\mathbf{x}}[M] \right) , \quad (13)$$

where $V[M]$ contains all gauge-invariant products of M up to $O(M^4)$ and $G^2 \rightarrow g^2/a^2$ as $a \rightarrow 0$. This is still not yet a finite number of operators, but we can obtain a finite set by further requiring a degree of *transverse locality* for operators that are products of gauge-invariant operators (see later).

This action in LF gauge reduces to

$$S(A_- = 0) = \int dx^0 dx^3 \sum_{\mathbf{x}} \left(\text{Tr} \left\{ \partial_+ M_r(\mathbf{x}) \partial_- M_r(\mathbf{x})^\dagger \right\} + \text{c.c.} + \frac{1}{G^2} \text{Tr} \left\{ (\partial_- A_+)^2 \right\} + \text{Tr} \left\{ A_+ J^+(\mathbf{x}) \right\} - V_{\mathbf{x}}[M] \right) , \quad (14)$$

where

$$J^+ = i \left(M_r(\mathbf{x}) \overleftrightarrow{\partial}_- M_r^\dagger(\mathbf{x}) + M_r^\dagger(\mathbf{x} - a\hat{\mathbf{r}}) \overleftrightarrow{\partial}_- M_r(\mathbf{x} - a\hat{\mathbf{r}}) \right) . \quad (15)$$

We note that the equation of motion for A_+ is a constraint (no x^+ -derivatives)

$$\frac{2}{G^2} (\partial_-)^2 A_+ = J^+ - \frac{1}{N} \text{Tr} J^+ , \quad (16)$$

which can be inverted for A_+

$$A_+ = \frac{G^2}{2} \frac{1}{\partial_-^2} \left(J^+ - \frac{1}{N} \text{Tr} J^+ \right) . \quad (17)$$

C Canonical LF Quantisation

The following canonical momenta can be straightforwardly derived in gauge-invariant form from the usual definition in terms of the energy momentum tensor $P^\alpha = \int dx^- \sum_{\mathbf{x}} T^{+\alpha}$;

²⁾ Because we use anti-periodic x^- boundaries, the $\int dx^- A_-$ zero mode has been removed from the theory. Its omission can be shown to renormalise the mass of the M field.

$$P^+ = 2 \int dx^- \sum_{\mathbf{x}} \text{Tr} \left\{ \partial_- M_r(\mathbf{x}) \partial_- M_r(\mathbf{x})^\dagger \right\} \quad (18)$$

$$P^- = \int dx^- \sum_{\mathbf{x}} V[M] - \frac{G^2}{4} \text{Tr} \left\{ J^+ \frac{1}{\partial_-^2} J^+ \right\} + \frac{G^2}{4N} \text{Tr} J^+ \frac{1}{\partial_-^2} \text{Tr} J^+ . \quad (19)$$

The transverse momenta \mathbf{P} and boost-rotations $M^{\mu\nu}$ will be discussed shortly (see ref. [2] for a summary of important properties of the Poincaré generators in light-front formalism). M_r is conjugate to $\partial_- M_r^\dagger$, and we impose equal- x^+ commutation relations

$$\left[M_{r,ij}(x^-, \mathbf{x}), \left(\partial_- M_{s,kl}(y^-, \mathbf{y}) \right)^\dagger \right] = \frac{1}{2} \delta_{il} \delta_{jk} \delta(x^- - y^-) \delta_{\mathbf{x},\mathbf{y}} \delta_{r,s} . \quad (20)$$

where $i, j \in \{1, \dots, N\}$ are colour indices.

It is useful to make a Fourier decomposition at $x^+ = 0$ with respect to x^- in terms of free fields

$$M_r(x^+ = 0, x^-, \mathbf{x}) = \frac{1}{\sqrt{4\pi}} \int_0^\infty \frac{dk^+}{\sqrt{k^+}} \left(a_{-r}(k^+, \mathbf{x}) e^{-ik^+ x^-} + a_r^\dagger(k^+, \mathbf{x}) e^{ik^+ x^-} \right) \quad (21)$$

that induces

$$\left[a_{\lambda,ij}(k^+, \mathbf{x}), a_{\rho,kl}^*(\tilde{k}^+, \mathbf{y}) \right] = \delta_{ik} \delta_{jl} \delta_{\lambda\rho} \delta_{\mathbf{x},\mathbf{y}} \delta(k^+ - \tilde{k}^+) , \quad (22)$$

$$\left[a_{\lambda,ij}(k^+, \mathbf{x}), a_{\rho,kl}(\tilde{k}^+, \mathbf{y}) \right] = 0 , \quad (23)$$

$\lambda, \rho \in \{\pm 1, \pm 2\}$, $a_{\lambda,ij}^* = (a_\lambda^\dagger)_{ji}$. Acting on a Fock vacuum $|0\rangle$ defined by

$$a_{\lambda,ij}(k^+, \mathbf{x})|0\rangle = 0 \quad \forall k^+, \mathbf{x}, i, j, \lambda , \quad (24)$$

$a_{\lambda,ij}^*(k^+, \mathbf{x})$ creates a link-parton on $(\mathbf{x}, \mathbf{x} + a\hat{\lambda})$ on the transverse lattice and carrying longitudinal momentum k^+ . (This mixed momentum-coordinate representation will be useful for displaying confinement shortly.) The general Fock space then splits into blocks of definite total P^+

$$\{a^*(P^+)|0\rangle, a^*(P^+ - k^+)a^*(k^+)|0\rangle, \dots\} . \quad (25)$$

States of definite \mathbf{P} will be constructed shortly. The above Fock basis can be used to write down a matrix representation of \hat{P}^- , which can then be diagonalised to obtain LF wavefunctions Ψ_{lc} .

D Physical States

The first key point to note at this stage is that, because of positivity $k^+ > 0$, P^- and P^+ contain no terms with only creation operators a^* . This means that

the Fock vacuum satisfies : $P^- : |0 \rangle = : P^+ : |0 \rangle = 0$. Of course, this is only true so long as M is a massive degree of freedom; this is the region of couplings we will be exploring for Lorentz covariance. Provided no tachyons appear in the physical spectrum of boundstates, $|0 \rangle$ is the physical vacuum of the theory (in the presence of our cutoffs).

Fock space is further simplified by confinement [13]. Since the operator corresponding to k^+ is ∂_- , a glance at (19) shows that eliminating A_+ has introduced small- k^+ singularities. These are cut-off by \mathcal{L} , but will blow up as $\mathcal{L} \rightarrow \infty$ for fixed a in general. The divergence is avoided if the LF charge Q associated to the current J^+ (15) vanishes

$$\lim_{k^+ \rightarrow 0} \int dx^- e^{ik^+ x^-} J^+ = 0 . \quad (26)$$

Only certain Fock states satisfy this condition, : $Q : |\text{Fock} \rangle = 0$. They are the states invariant under the residual x^- -independent gauge transformations

$$M_r(\mathbf{x}) \rightarrow V(\mathbf{x}) M_r(\mathbf{x}) V^\dagger(\mathbf{x} + a\hat{\mathbf{r}}) , \quad \partial_- V = 0 . \quad (27)$$

This will include, for example, closed loops of link-partons on the transverse lattice

$$\begin{aligned} & \text{Tr}\{a_\lambda^\dagger(P^+ - k^+, \mathbf{x}) a_{-\lambda}^\dagger(k^+, \mathbf{x})\} |0 \rangle , \\ & \text{Tr}\{a_\lambda^\dagger(P^+ - k_1^+ - k_2^+ - k_3^+, \mathbf{x}) a_\lambda^\dagger(k_1^+, \mathbf{x} + a\hat{\lambda}) a_{-\lambda}^\dagger(k_2^+, \mathbf{x} + 2a\hat{\lambda}) a_{-\lambda}^\dagger(k_3^+, \mathbf{x} + a\hat{\lambda})\} |0 \rangle , \\ & \text{etc..} \end{aligned}$$

This simple geometrical picture of the finite-energy, gauge-singlets is why we chose to work in transverse configuration space but longitudinal momentum space initially.

A remarkable property of the Fock space of connected closed loops of links³ is that it is finite-dimensional in a sector of fixed P^+ and \mathbf{P} for given cut-off \mathcal{L} . If we write the momentum of the i th parton as

$$k_i^+ = \frac{n_i P^+}{K} , \quad (28)$$

where $n_i \in \{1/2, 3/2, \dots, K - 1/2\}$ and K is an integer that plays the role of dimensionless cut-off, the Fock space is finite-dimensional because individual momenta are discrete and bounded and the number of partons is bounded (by $2K$). This is the basis of the numerical treatment known as DLCQ [19]. Matrices of finite dimension can be calculated and diagonalised, and the results extrapolated to large K .

³⁾ These are the relevant ones in the $N = \infty$ limit for example.

E Momentum Eigenstates

To test Lorentz covariance it will be important to have explicit forms for boundstate wavefunctions at non-zero momentum. So far we have constructed a Fock basis diagonal in \hat{P}^+ , with modes localised on transverse links. To obtain states of definite non-zero transverse momentum \mathbf{P} , we can take a gauge-singlet p -link shape, say, and translate it over the transverse lattice sites \mathbf{y} according to

$$\sum_{\mathbf{y}} e^{i\mathbf{P}\cdot(\mathbf{y}+\bar{\mathbf{x}})} \text{Tr}\left\{a_{\lambda_1}^\dagger(k_1^+, \mathbf{x}_1 + \mathbf{y}) a_{\lambda_2}^\dagger(k_2^+, \mathbf{x}_2 + \mathbf{y}) \cdots a_{\lambda_p}^\dagger(k_p^+, \mathbf{x}_p + \mathbf{y})\right\} |0\rangle . \quad (29)$$

We must have

$$\begin{aligned} \sum_{i=1}^p \hat{\lambda}_i &= 0 \\ \mathbf{x}_i &= \mathbf{x}_{i-1} + a \hat{\lambda}_{i-1} , \quad i < 1 \leq p \\ \sum_{i=1}^p k_i^+ &= P^+ \end{aligned} \quad (30)$$

and it is convenient to set the phase convention by adopting a momentum-weighted ‘centre-of-mass’

$$\bar{\mathbf{x}} = \frac{1}{P^+} \sum_{i=1}^p k_i^+ \left(\mathbf{x}_i + \frac{a \hat{\lambda}_i}{2} \right) . \quad (31)$$

This is consistent with the canonical gauge-invariant form of the M_{-r} component of the boost-rotation tensor

$$M_{-r} = 2 \int dx^- \sum_{\mathbf{x}} (x_r + a\delta_{rs}/2) \text{Tr}\left\{\partial_- M_s(\mathbf{x}) \partial_- M_s(\mathbf{x})^\dagger\right\} \quad (32)$$

which gives

$$e^{-ib^r M_{-r}} a_\lambda^\dagger(k^+, \mathbf{x}) e^{ib^r M_{-r}} = a_\lambda^\dagger(k^+, \mathbf{x}) e^{-ik^+ \mathbf{b} \cdot (\mathbf{x} + a\hat{\lambda}/2)} \quad (33)$$

$$e^{-ib^r M_{-r}} |\Psi_{\text{lc}}(P^+, \mathbf{P})\rangle = |\Psi_{\text{lc}}(P^+, \mathbf{P} - \mathbf{b}P^+)\rangle , \quad (34)$$

Thus, non-zero \mathbf{P} states in the Brillouin zone $(-\pi/a, \pi/a)$ can be conveniently obtained from those at $\mathbf{P} = 0$.⁴ Longitudinal boosts generated by

$$M_{-+} = 2 \int dx^- \sum_{\mathbf{x}} x^- \text{Tr}\left\{\partial_- M_r(\mathbf{x}) \partial_- M_r(\mathbf{x})^\dagger\right\} \quad (35)$$

⁴⁾ The construction of the operator \mathbf{P} itself is more problematic however, since the canonical expression is not gauge-invariant, while the simplest gauge-invariant extensions are not quadratic.

are even more trivial; up to unimportant global phases they simply rescale longitudinal momentum

$$e^{isM_{-+}} |\Psi_{\text{lc}}(P^+, \mathbf{P})\rangle = |\Psi_{\text{lc}}(e^{-s}P^+, \mathbf{P})\rangle . \quad (36)$$

Thus, longitudinal momentum fractions $x_i = k_i^+/P^+$ are Lorentz-boost invariant. A perfectly relativistic state should have a dispersion of the form

$$\mathcal{M}^2 = 2P^+P^- - \mathbf{P}^2 \quad (37)$$

and so diagonalising \hat{P}^- in a basis of fixed P^+ and \mathbf{P} is equivalent to finding the masses \mathcal{M} of boundstates. Because of the transverse lattice however, there will be corrections to (37) (see later) which we will try to minimize as part of the optimization of Lorentz covariance.

III HEAVY SOURCES AND WINDING MODES

Pure QCD is characterised by a single scale in terms of which all dimensionful quantities, such as hadron masses, can be expressed. A convenient measure of this scale is the string tension σ , the asymptotic slope of the linearly-rising confining potential between heavy sources. Moreover, by measuring this slope in both transverse lattice directions \mathbf{x} and the continuum space direction x^3 , and requiring it to be rotationally invariant, we fix the lattice spacing a in dimensionful units.

A Winding Modes

In the transverse directions, a more accurate measurement of σ can be made by examining winding modes rather than heavy sources. By making the transverse lattice compact in direction $\hat{\lambda}$ say

$$\mathbf{x} \equiv \mathbf{x} + a\hat{\lambda}D_\lambda , \quad (38)$$

where D_λ is the number of transverse links in direction λ , we can construct a basis of Fock states that wind around these directions, e.g.

$$\text{Tr}\left\{a_{\lambda_1}^\dagger(k_1^+, \mathbf{x}) a_{\lambda_2}^\dagger(k_2^+, \mathbf{x} + a\hat{\lambda}_1) \cdots a_{\lambda_p}^\dagger(k_p^+, \mathbf{x} + a\hat{\lambda}D_\lambda - a\hat{\lambda}_p)\right\} |0\rangle \quad (39)$$

has winding number 1 in direction $\hat{\lambda}$. The mass spectrum of such winding modes should rise linearly with D_λ with slope σ as $D_\lambda \rightarrow \infty$. Unlike the potential between heavy sources, there are no ‘endpoint’ effects, and so the asymptotic linear rise should set in more quickly, especially for the lowest mass eigenvalue

$$\mathcal{M} \rightarrow \sigma_T |\mathbf{n}| a \quad , \quad |\mathbf{n}| \rightarrow \infty \quad , \quad (40)$$

where $\mathbf{n} = (D_1 n_1, D_2 n_2)$ and n_1, n_2 are winding numbers in directions $(1, 0)$ and $(0, 1)$. (Note that we cannot construct winding modes around a compactified x^3 as this clashes with the choice of LF coordinates). The suffix ‘T’ indicates a transverse measurement of σ .

B Heavy Sources

To measure the potential between two heavy sources in transverse lattice gauge theory [20], we will start with a heavy scalar field $\phi(x^+, x^-, \mathbf{x})$ of large mass ρ in the fundamental representation of $SU(N)$. In addition to the pure glue ‘link-link’ interactions, we must now also consider interactions with ϕ ’s. The simplest are

$$\left(\partial_\alpha \phi^\dagger - i \phi^\dagger A_\alpha \right) (\partial_\alpha \phi + i A_\alpha \phi) \quad , \quad -\rho^2 \phi^\dagger \phi \quad (41)$$

such that ϕ couples to M via A_α .⁵ In LF gauge $A_- = 0$ this leads to the substitution

$$J^+ \rightarrow J_{\text{pure glue}}^+[M] + J_{\text{source}}^+[\phi] \quad (42)$$

$$(J_{\text{source}}^+)_{ij} = -i \partial_- \phi_i \phi_j^* + i \phi_i \partial_- \phi_j^* \quad (43)$$

Now for some kinematics: Let P_{full}^α represent the full 2-momentum of a system containing h heavy particles. It is convenient to split the full momentum into a “heavy” part plus a “residual” part P^α due to interactions,

$$P_{\text{full}}^\alpha = \rho h v^\alpha + P^\alpha \quad , \quad (44)$$

where v^α is the covariant velocity of the heavy sources, $v^\alpha v_\alpha = 1$. Note that P^α , as defined here, is not positive definite. The full invariant mass-squared (at $\mathbf{P} = 0$) is

$$\mathcal{M}^2 = 2P_{\text{full}}^+ P_{\text{full}}^- = (h\rho)^2 + 2h\rho v^+ P^- + 2P^+ (P^- + h\rho v^-) \quad (45)$$

The choice of v^+ is arbitrary and it is convenient to choose it such that $P^+ = 0$. Consequently, $v^+ P^-$ is just the shift of the full invariant mass \mathcal{M} due to the interactions:

$$\mathcal{M} = h\rho + v^+ P^- + O(1/\rho) \quad . \quad (46)$$

Thus, $v^+ P^-$ is the usual energy associated with the heavy quark potential.

⁵⁾ The full set of operators allowed at leading order of the colour-dielectric expansion is described in ref. [7], but we skip the details here.

C Fock Space

We define creation-annihilation operators associated with the heavy field:

$$\phi_i(x^+ = 0, x^-, \mathbf{x}) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} \frac{dk}{\sqrt{\rho v^+ + k}} \left(b_i(k, \mathbf{x}) e^{-i(v^+ \rho + k)x^-} + d_i^*(k, \mathbf{x}) e^{i(v^+ \rho + k)x^-} \right) \quad (47)$$

$$[b_i(k, \mathbf{x}), b_j^*(\tilde{k}, \mathbf{y})] = \delta_{ij} \delta_{\mathbf{x}, \mathbf{y}} \delta(k - \tilde{k}), \quad \text{etc.} \quad (48)$$

The $e^{i\rho v_\alpha x^\alpha}$ term removes an overall ρv^α from the 2-momentum. $b_i^*(k, \mathbf{x})$ creates a source particle at site \mathbf{x} carrying residual momentum k , while d^* creates an anti-particle. The residual gauge invariance in $A_- = 0$ gauge again leads to confinement into singlet states. An example of a gauge-invariant Fock state with two co-moving sources at \mathbf{x} and \mathbf{y} respectively, maintaining a fixed a x^3 -separation L is

$$\int_{-\infty}^{\infty} dl e^{2iLlv^-} b_i^*(l - P_{\text{link}}/2, \mathbf{x}) a_{ij}^*(k_1, \mathbf{x}) \cdots a_{mn}^*(k_p, \mathbf{y}) d_n^*(-l - P_{\text{link}}/2, \mathbf{y}) |0\rangle \quad (49)$$

where $\sum_{i=1}^p k_i = P_{\text{link}}$. To obtain LF eigenfunctions of \hat{P}^- we must take linear combinations of all possible $P_{\text{link}} > 0$. In practice this requires us to introduce a high-energy cut-off on P_{link} , in addition to the DLCQ one, K . $v^+ \hat{P}^-$ is then a finite dimensional matrix (at least for the connected Fock states that dominate in the large- N limit) whose eigenvalues may be extrapolated to infinite cutoff.

The lowest eigenvalue of $v^+ \hat{P}^-$ can be compared with the popular phenomenological form of the potential between heavy sources of spatial separation R

$$V(R) = \sigma R + c_1 + \frac{c_2}{R} \quad (50)$$

$$R = \sqrt{a^2 |\mathbf{n}|^2 + L^2} \quad \text{Source Separation,}$$

where $a\mathbf{n} = \mathbf{y} - \mathbf{x}$ and $L = y^3 - x^3$. For generic couplings in \hat{P}^- the eigenvalues will not simply be a function of R , i.e. σ, c_1, c_2 extracted from (50) will depend upon L and $|\mathbf{n}|$ separately. Once rotational invariance is restored, the value of σ can be set from experiment by the masses and decays of heavy mesons using $V(R)$ in a conventional Schrodinger equation. A typical value is $\sqrt{\sigma} \sim 440 \text{ MeV}$, though one must remember that this will includes effects of light-quarks which are neglected in pure gauge theory. Higher eigenvalues of $v^+ \hat{P}^-$ should correspond to hybrid heavy mesons.

D Setting the Scales

We now have everything necessary for performing a first-principles calculation. Let us now combine results from each sector — pure glue, winding, heavy sources.

Pure Glue

For generic couplings in \hat{P}^- , the glueball eigenstates at fixed momenta (P^+, \mathbf{P}) have eigenvalues which we may expand in powers of transverse momentum thus

$$2P^+P^- = G^2N \left(\mathcal{M}_0^2 + \mathcal{M}_1^2 a^2 |\mathbf{P}|^2 + 2\overline{\mathcal{M}}_1^2 a^2 P^1 P^2 + \mathcal{M}_2^2 a^4 |\mathbf{P}|^4 + \dots \right) . \quad (51)$$

Here we have chosen to factor out the coupling G^2 as an overall mass scale (G has units of energy). $\mathcal{M}_0, \mathcal{M}_1, \overline{\mathcal{M}}_2, \dots$ are then dimensionless numbers which we calculate when diagonalising \hat{P}^- . Eq.(51) is not, of course, in the form of a relativistic dispersion relation (37) in general. The coefficients, however, are functions of the couplings at our disposal and may be adjusted to regain a relativistic form.

Winding Modes

Let us write the lowest winding eigenvalue (40) as

$$2P^+P^- = a^2 \sigma_T^2 |\mathbf{n}|^2 = G^2 N \mathcal{W} |\mathbf{n}|^2 , \quad |\mathbf{n}| \rightarrow \infty . \quad (52)$$

Again, we factor out G^2 to set the scale, and \mathcal{W} is a dimensionless measured quantity dependent on the couplings.

Heavy Sources

Similarly, at fixed sources separation R

$$v^+P^- = \sigma R = G^2 N \mathcal{S} R , \quad R \rightarrow \infty . \quad (53)$$

\mathcal{S} is the dimensionless measured quantity that we can adjust with the couplings in \hat{P}^- .

The following conditions are necessary for Lorentz covariance of eigenstates

$$\sigma_T = \sigma = \text{constant} \quad (54)$$

$$\mathcal{M}_1^2 a^2 G^2 N - 1 = 0 \quad (55)$$

$$\overline{\mathcal{M}}_1 = 0 \quad (56)$$

Of course, there are further conditions that one could impose on solutions. But as we shall see in the next lecture, in practice these seem to be the most significant when trying to remove lattice discretization errors to a first approximation.

To test for (55) we need the dimensionless combination $a^2 G^2 N = \mathcal{W}/\mathcal{S}^2$ that follows from (52)(53). Combining (51)(53) allows us to express boundstate masses in units of the string tension

$$2P^+P^-(\mathbf{P}=0) = \mathcal{M}^2 = \frac{\mathcal{M}_0^2\sigma}{\mathcal{S}} \quad (57)$$

and also the lattice spacing itself $a = \sqrt{\mathcal{W}/\mathcal{S}\sigma}$.

IV EXAMPLE – LARGE- N QCD IN $2+1$ DIMENSIONS

We need to do a test calculation that can be accurately compared with known results and is relevant to non-abelian gauge theories in $3+1$ dimensions. $SU(\infty)$ gauge theory in $2+1$ dimensions provides an ideal test. It is physically very similar to $SU(3)$ pure gauge theory in $3+1$ dimensions, having dynamically-generated linear confinement and a discrete spectrum of glueballs; it is even quantitatively similar. But factorization of colour Traces in the Fock space states in the large- N limit and having only one transverse dimension make an accurate calculation feasible. Moreover, good Euclidean lattice Monte Carlo (ELMC) data has recently become available for this problem [21], which we can take as ‘the right answer’.⁶ The results presented here have been obtained with B. van de Sande and are new or improvements on our previously published work [5].

The discussion of the previous lectures is trivially adapted to $2+1$ dimensions. Here $x^\pm = (x^0 \pm x^2)/\sqrt{2}$ etc. and x^1 is the single transverse lattice dimension. Up order order M^4 , we have in the large N limit with G^2N finite (suppressing transverse indices for clarity)

$$\begin{aligned} P^- = & \int dx^- - \frac{G^2}{4} \text{Tr} \left\{ J^+ \frac{1}{\partial_-^2} J^+ \right\} + \frac{G^2}{4N} \text{Tr} J^+ \frac{1}{\partial_-^2} \text{Tr} J^+ + \mu^2 \text{Tr} \{ M M^\dagger \} \\ & + \frac{\lambda_1}{aN} \text{Tr} \{ M M^\dagger M M^\dagger \} + \frac{\lambda_2}{aN} \text{Tr} \{ M M M^\dagger M^\dagger \} + \frac{\lambda_3}{aN^2} \left(\text{Tr} \{ M M^\dagger \} \right)^2 \end{aligned} \quad (58)$$

where, in anticipation of always taking the $\mathcal{L} \rightarrow \infty$ limit, we have also used power-counting in longitudinal coordinates to limit the number of terms. There are in principle an infinite number of different products of individually gauge-invariant operators (like the λ_3 term), obtained by separating the operators in the transverse direction. We have also assumed a degree of transverse locality by keeping only the most local product to a first approximation. This approximation, like the colour-dielectric expansion in gauge-invariant powers of M , can of course be systematically investigated.

To maintain a trivial LF vacuum, we shall be considering the region $\mu^2 > 0$ with tachyon-free physical spectrum. G has dimensions of energy and will be

⁶⁾ The calculations of Teper are based on the traditional Wilson Euclidean lattice path integral formulation of gauge theory. The large N calculations were in fact performed about the same time as the light-front work, and to some extent motivated by it.

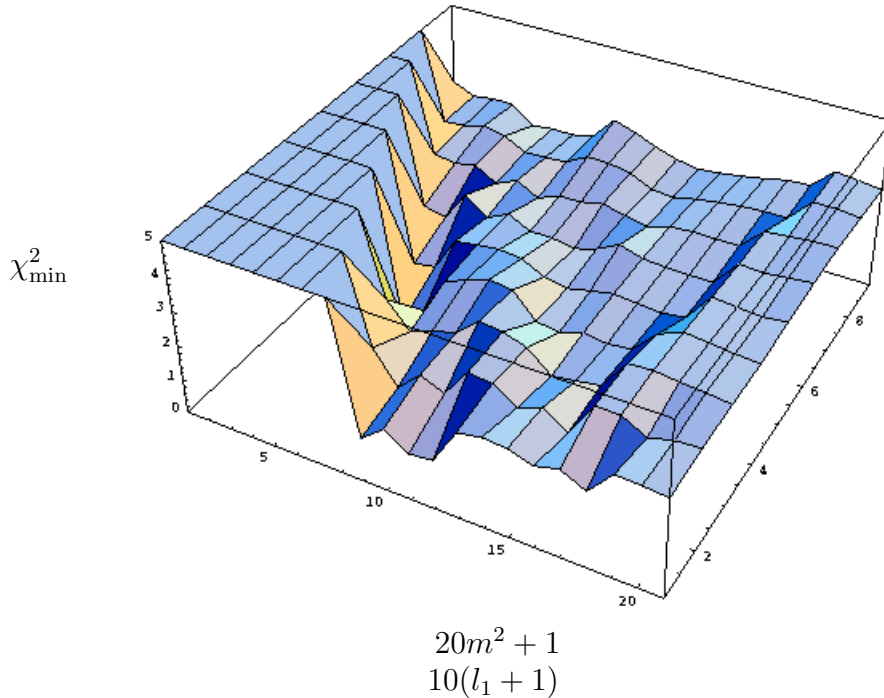


FIGURE 1. χ^2 -chart for l_1 .

used to express dimensionful quantities; it is convenient to form dimensionless versions of all the other couplings

$$m^2 = \frac{\mu^2}{G^2 N} , \quad l_i = \frac{\lambda_i}{a G^2 N} . \quad (59)$$

The basic technique we follow is to search the space $\{m, l_1, l_2, l_3\}$ for a one parameter trajectory on which observables show enhancement of Lorentz covariance — a ‘Lorentz trajectory’. Moving along this trajectory should correspond to changing the spacing a , eventually taking us to the continuum limit. We will, in fact, be prevented from reaching $a = 0$ by our self-imposed restriction $m^2 > 0$. Nevertheless, assuming gauge and Lorentz covariance *define* pure QCD, observables should be invariant along the (exact) Lorentz trajectory.⁷ Of course, we can only obtain some approximation to such a trajectory with a finite number of couplings.

To get some intuition for the space of couplings, let us construct charts that display the ‘size’ of violations of Lorentz covariance. We can use a χ^2 -test

⁷⁾ The idea is similar to the renormalised trajectory in the renormalisation group associated to a fixed point with one relevant direction. However we are not performing any explicit RG transformations here!

$$20m^2 + 1.$$

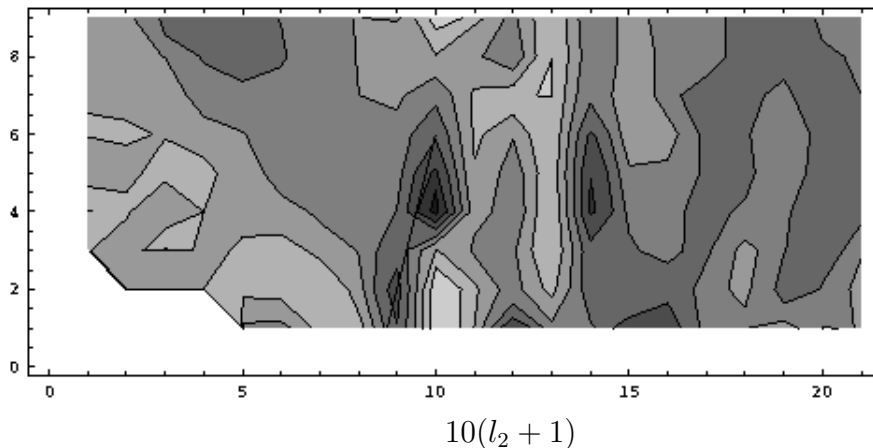


FIGURE 2. χ^2 -chart for l_2 . Darker shades correspond to lower χ^2_{\min} .

based on the variables (54))(55) (condition (56) is absent in 2+1-dimensions). Condition (55) in fact provides a number of variables, one for each boundstate (glueball) in the low-lying spectrum. To begin with, let us set the lowest glueball mass to the known value $\mathcal{M} = 4.05\sqrt{\sigma}$ [21], and try to get an isotropic speed of light (55) for this and the heavier low-lying glueballs. With a particular choice of variance assignments in the χ^2 -test, we can make plots of $\{\chi^2_{\min}, m^2, l_i\}$ for each l_i , where χ^2_{\min} is the minimum value of χ^2 with respect to the couplings except m^2 and l_i . For technical reasons l_3 almost completely decouples (we can set it to almost any fixed large value, see ref. [5]) so we only display charts for l_1 (fig.1) and l_2 (fig.2).

They each show an unique, narrow valley, running from small to large m^2 , at the bottom of which Lorentz covariance is optimised. One finds that changing the details of the χ^2 test changes these charts somewhat, *except* in the neighborhood of the valley in each case. Thus one finds a degree of universality in the results. If there were no Lorentz trajectory, one would not expect to obtain a robust valley. Having roughly located the candidate Lorentz trajectory, one can increase the resolution in its neighborhood, and perform more efficient iterative searches for the bottom of the valley.

In a first-principles calculation we cannot use the result for the lightest glueball mass in σ units — this should be predicted by the method. Performing a new χ^2 search in the neighborhood of the valley, but now including a variable to test (54), one finds a similar position for the valley bottom, without having to input the lightest glueball mass. The χ^2 per degree of freedom is always about 1 near the Lorentz trajectory, as it should be for reasonable criteria. Results as one moves along the valley bottom are displayed in fig.3 and fig.4.

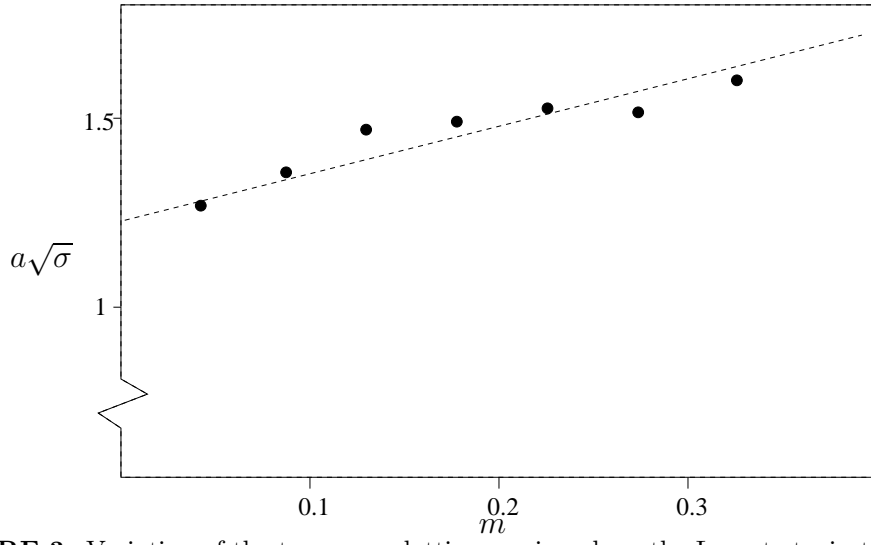


FIGURE 3. Variation of the transverse lattice spacing along the Lorentz trajectory. The fit is $1.275m + 1.23$.

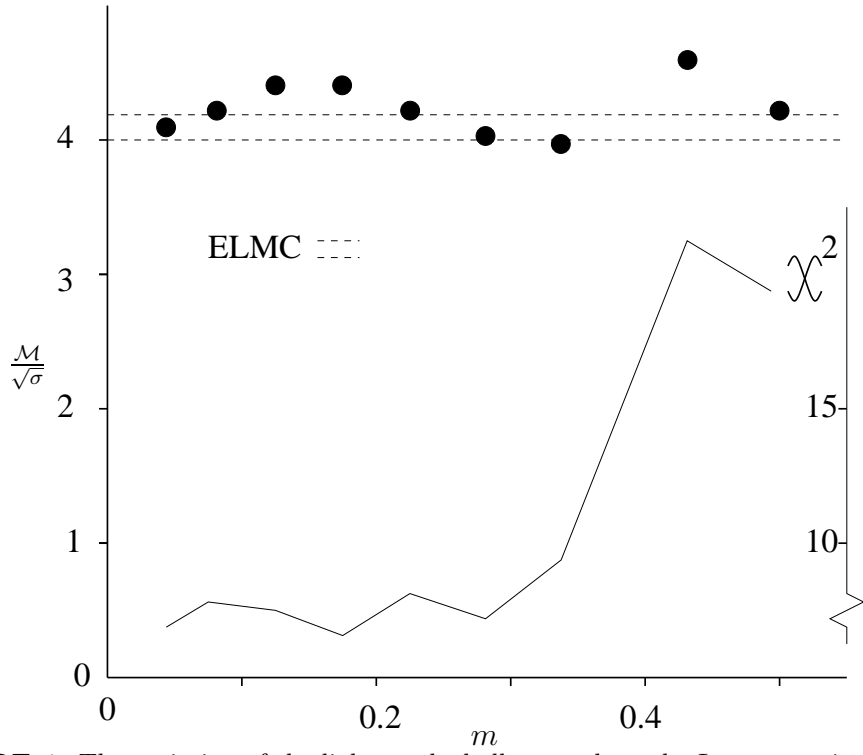


FIGURE 4. The variation of the lightest glueball mass along the Lorentz trajectory (together with the associated variation of the χ^2).

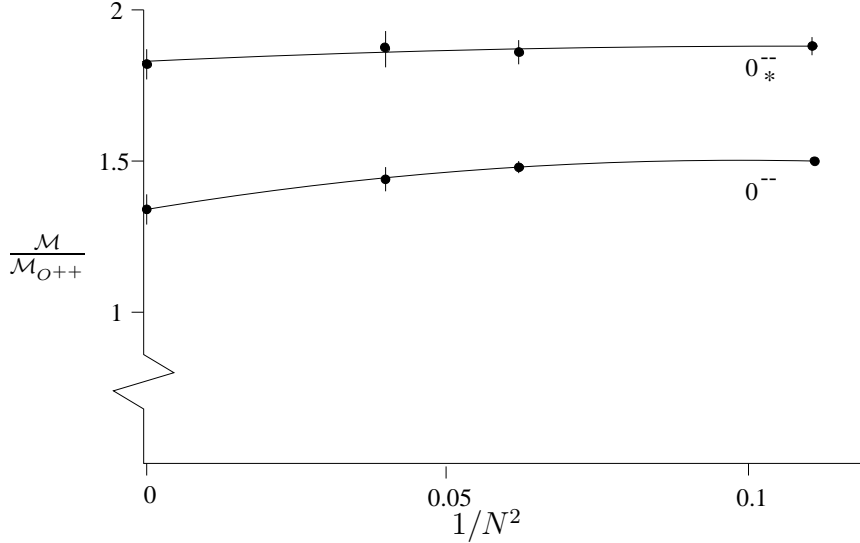


FIGURE 5. Variation (or lack of it!) of the glueball mass ratios $0^{--}/0^{++}$ and $0_*^{--}/0^{++}$ with N . The fits are $1.349 + 2.914/N^2 - 13.968/N^4$ for 0^{--} and $1.824 + 1.274/N^2 - 7.143/N^4$ for 0_*^{--} . The error on the $N = \infty$ results is from extrapolation to $\mathcal{L} = \infty$.

We see that the lattice spacing gradually decreases with m^2 as one moves along the Lorentz trajectory, but never becomes zero for $m^2 > 0$. Despite fluctuations, the lightest glueball mass scales in a way roughly consistent with the correct continuum answer.

Although the lightest glueball (a $J^{PC} = 0^{++}$) is behaving covariantly along the Lorentz trajectory — the speed of light deduced from the 0^{++} 's dispersion is isotropic to within % 2-3 — fluctuations are still present in fig.4. These are mostly due to the difficulty in accurately establishing the scale σ . Most of the fluctuations tend to cancel if we consider ratios of glueball masses, and good scaling is obtained along the Lorentz trajectory [5]. For higher glueballs with nearly-covariant wavefunctions, such as the 0^{--} and its excited state 0_*^{--} , this provides rather accurate mass ratio determinations in the large- N limit; see figure 5. Finite- N results from conventional ELMC in ref. [21] were fit to $A + B/N^2$, at confidence levels of order %50 – 70. Including our $N = \infty$ data, and fitting to $A + B/N^2 + C/N^4$, improves this to %95 for $0_*^{--}/0^{++}$ and > %99 for $0^{--}/0^{++}$ glueball mass ratios!

The heavy-source potential calculated at a point on the Lorentz trajectory is displayed in fig.6. It shows good restoration of spatial symmetry. The potential in the continuum spatial direction x^2 is a fit to

$$v^+P^- = 0.154LG^2N + 0.183\sqrt{G^2N} - \frac{0.178}{L} \quad (60)$$

at the point on the Lorentz trajectory of lowest overall χ^2 . One must be careful when interpreting (60) since the Coulomb potential in $2 + 1$ dimensions is

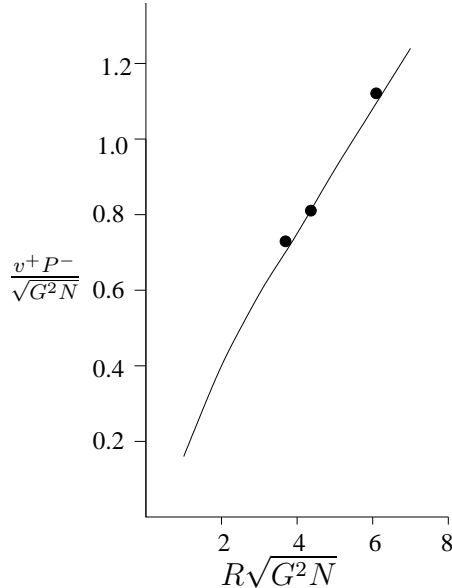


FIGURE 6. The heavy-source potential. Solid line is fit to potential for sources with x^2 -separation only; data points are values at one-link transverse separation and x^2 -separation $L\sqrt{G^2 N} = 0, 2.5, 5$.

logarithmic. The form (60) should be appropriate except at the very smallest L , where Coulomb corrections are expected. The $1/L$ term is a universal correction expected on the grounds of models of flux-string oscillations [22]. Universality implies that its coefficient should be invariant along the Lorentz trajectory. In reality, we find that it drifts slowly, a symptom that our approximation to the Lorentz trajectory is not an exact scaling trajectory and/or the form (60) is not sufficient to fit the potential. The coefficient 0.178 at the lowest χ^2 is nevertheless close to the theoretical value $d\pi/24$, where d is the number of transverse dimensions in which an ideal thin-string can oscillate ($d = 1$ here). But it is larger by 40% (0.178 gives $d \sim 1.4$). We found the same excess by another method to measure the ‘central charge’ $d + 2$, using the density of glueball states [4]. In ref. [4] we observed that this excess in the central charge comes from longitudinal degrees of freedom in the large- N QCD flux string, a mode of oscillation which an ideal thin-string does not possess.

One may wonder whether results really do improve as more operators are added to the hamiltonian. For a given χ^2 test, adding more operators to the hamiltonian will inevitably bring one closer to the Lorentz trajectory as measured by this test. We see from fig. 7 that the masses of glueballs, which are not part of the test, rapidly approach the correct values as more operators are added and covariance is improved.⁸

⁸⁾ The 2^{-+} state appears to overshoot the correct mass. It is the only state whose wave-function actually becomes less covariant when λ_1 is turned on.

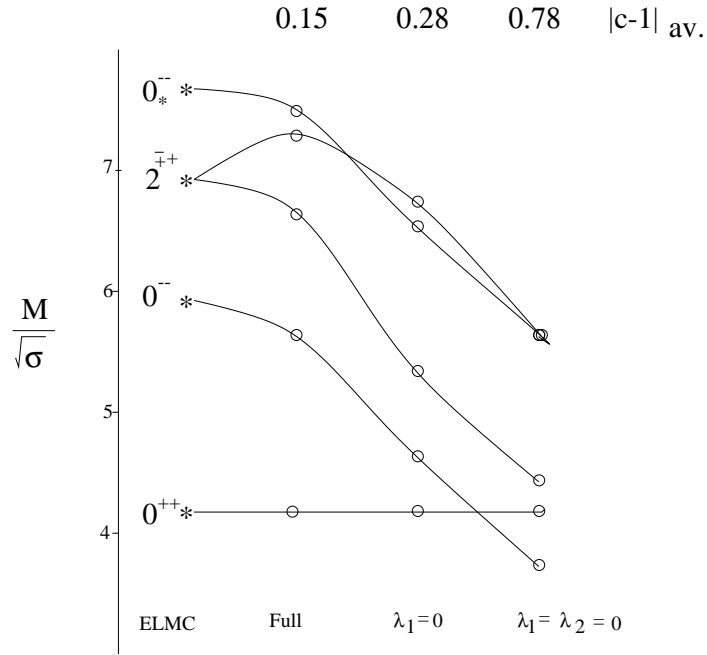


FIGURE 7. Variation of glueball mass ratios to the lightest mass (0^{++}), as more operators are included in the hamiltonian. $|c-1|_{\text{av}}$ measures the average deviation from 1 of the speed of light in the transverse direction, for the displayed glueballs.

A Future Work

The transverse lattice idea is already quite old [13]. The breakthrough, that has been made over the last year or so, was to show that in $2 + 1$ and $3 + 1$ dimensions large- N gauge theory exhibit a unique Lorentz covariant scaling trajectory on coarse transverse lattices [5–7]. Glueball masses on this trajectory are consistent with known results. Thus there is good reason to believe that the Lorentz-invariant wavefunctions obtained with this method are accurate also. They are completely new results, essentially unobtainable within any other quantisation scheme. It will be interesting to see what implications they have for experiment. The next step towards this goal is to couple these pure gauge theories to propagating quarks [13,23], and then enforce the Lorentz and chiral symmetries that define QCD to determine the couplings of the hamiltonian. We also need to understand better how to extrapolate results into the small a region. This is important not only formally for the existence of the continuum limit, but to compare calculations of hadronic structure with hard process experiments where the usual factorization theorems simplify the analysis. We are only at the beginning of the development of this subject, and it is difficult to know how far it will progress, but there does appear to be too much truth in it to ignore.

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